Section 11-1, Mathematics 104

Exponential functions

A famous story:

Supposedly a king asked one of his subjects to invent a new and interesting game. He invented the game of chess.

The king liked the game and asked the subject to name any reward he wanted. The subject requested that starting on one square of the chess board he would get 1 grain of rice. On the second 2, the third 4 and so forth, doubling each time. The king thought this was very reasonable.

Later the kind found that to provide his subject with all the rice he asked for, or its value in money, it would bankrupt the kingdom, so instead he had the subject executed.

Let's take a look. The number of grains requested is as follows:

$$f(x) = \sum_{n=0}^{63} 2^x = 2^{64} - 1$$

How big is this number?

Approximately 18,000,000,000,000,000,000 or 18 quintillion grains of rice.

The total wealth of the world is about $300,000,000,000,000, or 300 trillion dollars.

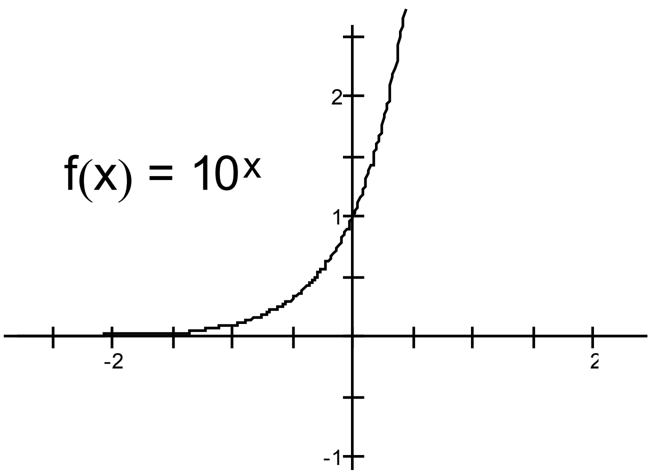
So if rice is worth more than $1 per 60,000 grains, this is more money that there is in the world.

Exponential functions can be like this.

An exponential function is one where there is a variable as an exponent and the base is a number greater than one.

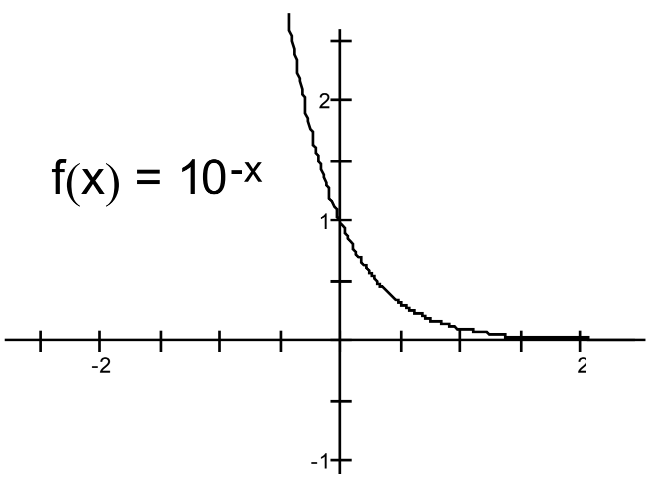
Example:

$f(x) = 10^x$



An exponential function can be an increasing or decreasing depending on the sign of the exponent.

$f(x) = 10^{-x}$



Increasing exponential functions eventually get larger than any polynomial function.

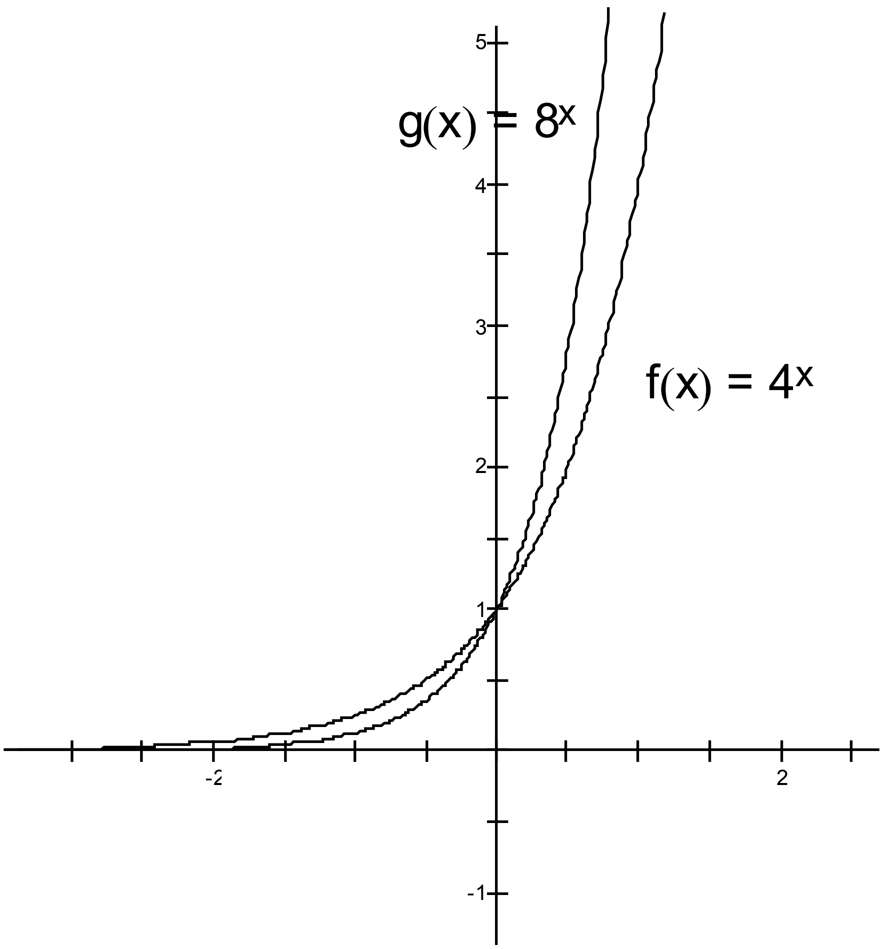
For all x large enough $1.000001^{x/1000} > x^{1000}$

The larger the base, the more quickly an exponential function will increase, or decrease.

Example: Compare

$f(x) = 4^x$

$f(x) = 8^x$



The number *e*.

There is a number in mathematics almost as important as π.
The number is given the name *e*.
It is sometimes known as Euler's constant.

$e \approx 2.718281828....$

Like π it is irrational and its decimal representation never repeats.

If you look on your calculator, you may see an *e* button, or the exponential function e^x or the *ln*() function.

ln is short for The Natural Log. We will look at logs in a couple of days, but a log function has a number which is its base, and the natural log has *e* as its base.

e shows up in a number of places in mathematics, the first to look at is compounded interest calculations.

Compound interest

If you are getting compound interest from your bank, you may know that the compounding period can vary.

Example:

You put $1 in the bank at 100% interest.

The formula for compound interest is

$$B = P\left(1+\frac{r}{n}\right)^{nt}$$

B is the balance after t years

P is the principle or starting amount

r is the yearly interest rate

n is the number of compounding periods in a year.

Here is what happens if you compound it at different periods:

Yearly(1)	Monthly	Daily	Hourly
1	1	1	1
2.0	2.613	2.715	2.718

If we keep decreasing the period, the amount in our bank account after 1 year gets closer and closer to the number e.

If we want to compound using infinitely small compounding periods, called continuous compounding we can use the formula

$$B = Pe^{rt}$$

Example:

Find the balance after 6 years of an investment of $15,000 at 8% compounded at these periods:

Quarterly

Montly

Continously

$$B_q = 15,000\left(1+\frac{.08}{4}\right)^{4\cdot 6} = 24,126.57$$

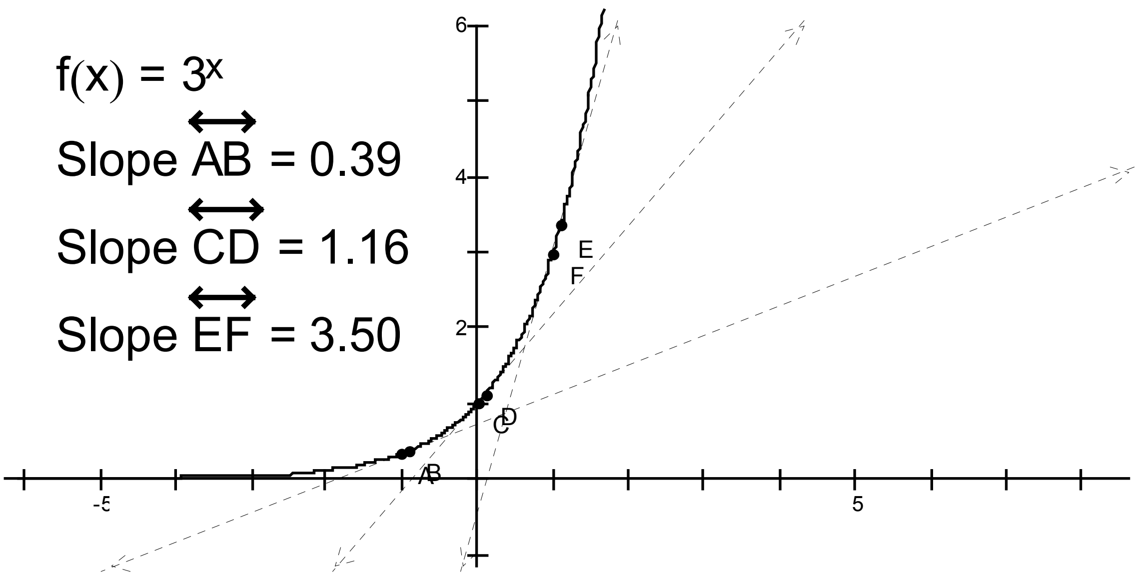
$$B_m = 15,000\left(1+\frac{.08}{12}\right)^{12\cdot 6} = 24,202.53$$

$$B_c = 15,000e^{.08\cdot 6} = 24,241.12$$

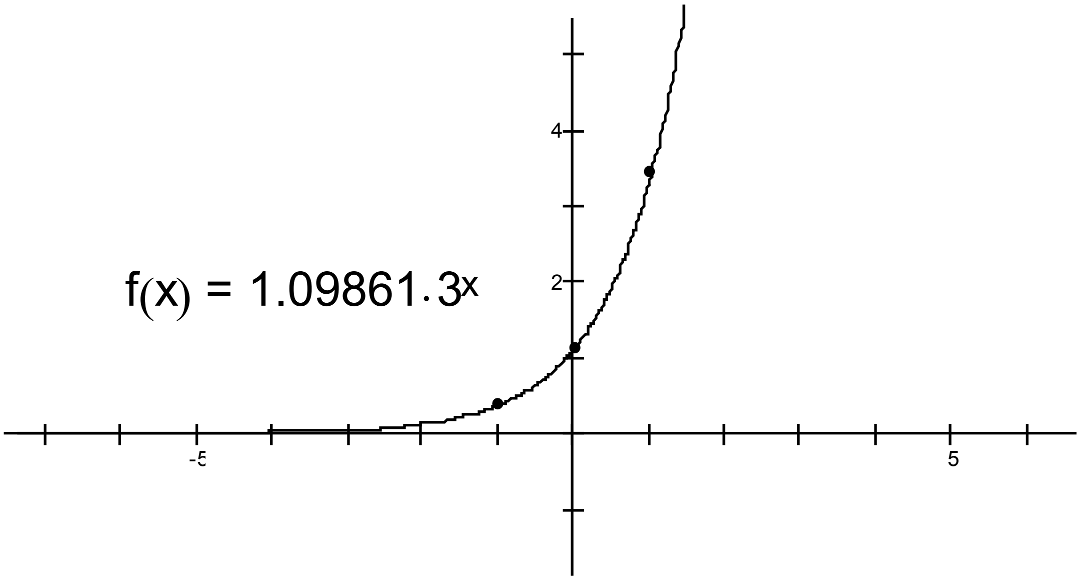
If time, some *e* magic

There is another place that *e* shows up.

If you take an exponential function, and you choose pairs of points on the function that are close together, you can find the slope of these points and graph them.



If you then plots these points you get a surprising result.



It's another exponential function.

This new function is not the same as the original function.

Well this is true for all bases except 1, the base *e*.

So it seems that *e* has some mysterious importance in math, even though the number seems quite arbitrary.